CURSO 2025-2: FUNCTIONS OF BOUNDED VARIATION AND ISOPERIMETRIC PROBLEMS

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6CP, martes y miercoles 10:00 - 11:30

1. Course Overview

The classical isoperimetric theorem states:

> Among all sets $E \subset \mathbb{R}^2$ with $\text{Area}(E) = 1$, the one with the smallest *perimeter* is the disc.

Despite being identified some 2500 years ago, it took until 1901 for Hurwitz to provide a rigorous proof under the assumption that the sets E are bounded by Jordan curves.

Still, some natural questions arise:

- What about higher dimensions?
- What is the largest possible class of measurable sets for which we can define some notion of 'perimeter'?
- Is the 'perimeter' always the same as the 'boundary'?
- How do we measure the size of this 'perimeter'?

These questions were answered in the 1950s when De Giorgi gave his brilliant solution to the Isoperimetric Problem. The heart of this solution is the study of functions of Bounded Variation. Roughly speaking, a function f is said to have *bounded variation* if its weak derivatives are Radon measures. Functions of Bounded Variation form the basic language for many so-called "free discontinuity problems" in the Calculus of Variations such as Mumford-Shah image segmentation problems, fracture and plasticity in continuum mechanics, edge detection in image analysis, etc. Moreover, the ideas in De Giorgi's solution to the isoperimetric problem have been extremely influential in Geometric Measure Theory and the study of regularity in geometric Variational Problems such as minimal surfaces.

The purpose of this class is to cover De Giorgi's solution to the Isoperimetric Problem as an introduction to key ideas in Geometric Measure Theory and the geometric problems in the Calculus of Variations.

2. Topics

(1) Basic results from measure theory
  a. Properties of Radon measures in $\mathbb{R}^n$
  b. Rademacher's Theorem

c. Area and Coarea Formulae

(2) A review of Sobolev Spaces
(3) Functions of Bounded Variation and Sets of Finite Perimeter
   a. Basic properties of BV functions
   b. The Jump Set and the reduced boundary
   c. The Gauss-Green Theorem
(4) The Isoperimetric Problem
   a. Steiner Symmetrization
   b. The Direct Method of Calculus of Variations
(5) Partial Regularity of perimeter almost-minimizers
   a. Local regularity
   b. Analysis of singularities

3. Prerequisites

THIS COURSE WILL BE TAUGHT IN ENGLISH. Further, it will assume familiarity with the Basic Courses: Análisis Real I and Análisis Funcional I. Students from "Métodos de análisis matemático para la resolución de problemas variacionales y ecuaciones en derivadas parciales" are highly encouraged to enroll.

4. Bibliography

(1) Evans, L. C., and Gariepy, R. F.. *Measure Theory and Fine Properties of Functions, Revised Edition*. Reino Unido, CRC Press; 2015.
(2) Maggi F. *Sets of Finite Perimeter and Geometric Variational Problems: An Introduction to Geometric Measure Theory*. Cambridge University Press; 2012.
(3) Ambrosio, L., Fusco, N., and Pallara, D.. *Functions of Bounded Variation and Free Discontinuity Problems*. Oxford University Press; 2000.

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