CURSO 2026-1: REGULARITY THEORY FOR PERIMETER MINIMIZERS UNDER VOLUME CONSTRAINT

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9CP, Lunes, Miercoles, y Vienres 10:00 - 11:30am

1. Course Overview

Consider the *Relative Isoperimetric Problem*:

> Given an open set $\Omega \subset \mathbb{R}^n$, among all $\mathcal{L}^n$-measurable sets which satisfy $\mathcal{L}^1(E \cap \Omega) = 1$, do perimeter-minimizers exist? What can be said about them?

Fortunately, existence of perimeters-minimizers is given by the standard machinery of sets of finite perimeter. This leaves the question: what do these minimizers look like? If we choose $\Omega = \mathbb{R}^n$, this is the *Isoperimetric Problem*, and we know that for all $n \in \mathbb{N}$ the unique (up to translation) solution is the ball. But, if $\Omega \neq \mathbb{R}^n$, then perimeter-minimizing sets will depend on Ω. In general, they will not be a ball!

To study perimeter-minimizers in the relative case, we focus upon the local structure. We say that $x \in \partial E$ is a *regular point* if the perimeter measure $||\partial E||$ has a flat measure-theoretic tangent at x. We say that x is a *singular point* if there are no flat measure-theoretic tangents to $||\partial E||$ at x. The main questions are:

- If $x \in \partial E$ is a regular point, can we write $\partial E \cap B_r(x)$ as the graph of a function for small enough $0 < r$? If so, how smooth will that function be?
- Can there be singular points? If so, what structure does the set of singular points have?

This course will cover the theory which is used to answer these questions for (almost) perimeter-minimizers of Relative Isoperimetric Problems. These techniques have been extremely important for the study of the regularity and structure of the singular set for many geometric problems. For example, versions are used in the study of regularity of minimal hypersurfaces, regularity for varifolds in arbitrary co-dimension, regularity of the free boundary in certain Free Boundary Problems, among other applications.

2. Topics

(1) Some background on Sets of Finite Perimeter
   a. Definition and basic properties of Sets of Finite Perimeter
   b. Review of basic results
   c. Set Operations

(2) The Calculus of Variations

   a. Minimizers, Local Minimizers, and Local Almost Minimizers.
   b. Comparison sets
   c. First Variation Formula
   d. Second Variation Formula
(3) Slicing Formulas
   a. Slicing $(n-1)$-rectifiable sets
   b. Slicing perimeters by hyperplanes
(4) The Regularity Theory (i.e., how to do PDEs with sets)
   a. Tilt-excess and Height-excess.
   b. Lipschitz Approximation
   c. Reverse Poincare Inequality
   d. Harmonic Approximation
   e. The Iteration argument and local $C^{1,\alpha}$-regularity.
(5) Structure of the Singular Set
   a. Blow-ups and tangent cones
   b. Simon's Theorem
   c. Federer's Dimension Reduction Argument
   d. Dimensional estimates on the Singular Set.

If time permits, we may also cover related topics.

3. Prerequisites

THIS COURSE WILL BE TAUGHT IN ENGLISH. Familiarity with Functions of Bounded Variation and sets of finite perimeter is highly encouraged.

4. Bibliography

(1) Maggi F. *Sets of Finite Perimeter and Geometric Variational Problems: An Introduction to Geometric Measure Theory.* Cambridge University Press; 2012.
(2) Giusti, E. *Minimal Surfaces and Functions of Bounded Variation.* Birkhäuser; 1984.

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